

Existence of perpetual points in nonlinear dynamical systems and its applications

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A new class of critical points, termed as *perpetual points*, where acceleration becomes zero but the velocity remains non-zero, are observed in dynamical systems. The velocity at these points is either maximum or minimum or of inflection behavior. These points also show the bifurcation behavior as parameters of the system vary. These perpetual points are useful for locating the hidden oscillating attractors as well as co-existing attractors. Results show that these points are important for better understanding of transient dynamics in the phase space. The existence of these points confirms whether a system is dissipative or not. Various examples are presented, and results are discussed analytically as well as numerically.

Keywords: Perpetual points; Stability and bifurcation; Hidden attractors; Conservation

1. Introduction

The stationary points of any dynamical system are the ones where velocity and acceleration of the system simultaneously become zero. These points, which are important to understand the system in the first step, are termed as fixed points (FP). The stability of these fixed points and the motion around them have been studied in detail, and can be found in any standard book on dynamical systems [Ott, 1993; Strogatz, 1994; Jordan & Smith, 2009; Kuznetsov, 2004; Kaushal & Prashar 2008; Lakshmanan & Rajasekar, 2003; Arrowsmith & Place, 1990].

In this paper we show that, in nonlinear dynamical systems, there also exist a new class of points (other than FPs) where acceleration becomes zero while velocity remains nonzero. These points can be termed as *perpetual points* (PP)¹. These points can be traced by considering the higher derivative of the velocity vector of the system. We demonstrate here that a study of these perpetual points is necessary for understanding several new features of dynamical systems. In particular, these points can be useful for locating the attractors, understanding the dynamics in phase space, and confirming whether a system is conservative or not. To this effect, several examples of different type of dynamical systems are analyzed here: ranging from low to high dimension, as well as from simple to complex ones. We also observe that these points show the bifurcation behavior as parameters are changed.

This paper is organized as follows. We study the linear stability around the perpetual points in next section. The distinct features of PPs and FPs are discussed in Sec. 3. This is followed by bifurcation analysis of PPs in Sec. 4. The use of PPs are discussed in Sec. 5. The results are summarized in Sec. 6.

¹ The name “perpetual” is used in analogy with “perpetual motion” which describes a motion that continues indefinitely without any external driving force (acceleration) – see http://en.wikipedia.org/wiki/Perpetual_motion

2. Perpetual points and stability analysis

Consider a general dynamical system specified by the equations

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, \alpha) \quad (1)$$

where $\mathbf{X} = (x_1, x_2, \dots, x_n)^T$ is n -dimensional vector of dynamical variables and $\mathbf{F} = (f_1(\mathbf{X}), f_2(\mathbf{X}), \dots, f_n(\mathbf{X}))^T$ specifies the evolution equations (velocity vector) of the system with internal parameter α . Here T stands for transpose of the vector. The first order Taylor expansion due to a small perturbation, $\delta\mathbf{X}$ ², in Eq. (1) around the fixed point \mathbf{X}_{FP} (where $\dot{\mathbf{X}} = 0$)³ leads to

$$\delta\dot{\mathbf{X}} = \mathbf{F}_{\mathbf{X}^T}(\mathbf{X}, \alpha)|_{\mathbf{X}_{FP}} \cdot \delta\mathbf{X} \quad (2)$$

where $\mathbf{F}_{\mathbf{X}^T}(\mathbf{X}, \alpha)$ is Jacobian of size $n \times n$. The dynamics near the fixed points depend on the eigenvalues (l_i) and the corresponding eigenvectors of this matrix, and the same has been well studied and documented in literature [Ott, 1993; Strogatz, 1994; Jordan & Smith, 2009; Kuznetsov, 2004; Kaushal & Prashar 2008; Lakshmanan & Rajasekar, 2003; Arrowsmith & Place, 1990].

Acceleration of the system can be obtained by taking derivative of Eq. (1) with respect to time⁴, viz.

$$\begin{aligned} \ddot{\mathbf{X}} &= \mathbf{F}_{\mathbf{X}^T}(\mathbf{X}, \alpha) \cdot \mathbf{F}(\mathbf{X}, \alpha) \\ &= \mathbf{G}(\mathbf{X}, \alpha) \end{aligned} \quad (3)$$

where $\mathbf{G} = \mathbf{F}_{\mathbf{X}^T}(\mathbf{X}, \alpha) \cdot \mathbf{F}(\mathbf{X}, \alpha)$ may be termed as acceleration vector. For convenience, we drop \mathbf{X} and α from the arguments of \mathbf{F} and \mathbf{G} from now on.

For zero acceleration of the system, we set $\ddot{\mathbf{X}} = \mathbf{G} = 0$ in Eq. (3) which gives a set of points where velocity $\dot{\mathbf{X}}$ may be either zero or a nonzero constant. This set includes the fixed points (\mathbf{X}_{FP}) with zero velocity as well as a subset of new points with nonzero velocity⁵. These nonzero velocity points are the perpetual points (\mathbf{X}_{PP})¹

The linear stability analysis of Eq. (3) around the perpetual points gives

$$\begin{aligned} \delta\ddot{\mathbf{X}} &= (\mathbf{I} \otimes \delta\mathbf{X}^T) \cdot \mathbf{F}_{\mathbf{X}\mathbf{X}^T} \cdot \mathbf{F} + \mathbf{F}_{\mathbf{X}^T} \cdot \mathbf{F}_{\mathbf{X}^T} \cdot \delta\mathbf{X} \\ &= \mathbf{G}_{\mathbf{X}^T}|_{\mathbf{X}_{PP}} \cdot \delta\mathbf{X} \end{aligned} \quad (4)$$

where \otimes is direct product while \mathbf{I} and $\mathbf{F}_{\mathbf{X}\mathbf{X}^T}$ are identity and Hessian matrices of dimension $n \times n$ and $nn \times n$ respectively. Note that matrix $\mathbf{G}_{\mathbf{X}^T}$ contains both the Jacobian ($\mathbf{F}_{\mathbf{X}^T}$) and Hessian ($\mathbf{F}_{\mathbf{X}\mathbf{X}^T}$) matrices. For simplicity, the ij -element of matrix $\mathbf{G}_{\mathbf{X}^T}|_{\mathbf{X}_{PP}}$ can be written as

$$(\mathbf{G}_{\mathbf{X}^T})_{i,j} = \sum_{k=1}^n [f_k f_{ix_k x_j} + f_{kx_j} f_{ix_k}]. \quad (6)$$

The general solution of the second order differential equation, Eq. (5) around the perpetual points for nonzero eigenvalues can be written as

$$\delta\mathbf{X} = \sum_i^n \mathbf{V}_i [c_{i1} \exp(+\sqrt{\mu_i}t) + c_{i2} \exp(-\sqrt{\mu_i}t)] \quad (7)$$

where μ_i and \mathbf{V}_i are the eigenvalues and eigenvectors respectively [Taylor, 2011]. Here c_{i1} and c_{i2} are the constants which depend on the initial position $\mathbf{X}(0)$ and perturbation $\delta\mathbf{X}(0)$. This solution determines the variation of velocity around the perpetual points. Note that the solution of equation Eq. (7) and Eq. (2) are not the same as acceleration vector², Eq. (3), may not be invertible to velocity vector, Eq. (1). The properties of these perpetual points, using Eqs. (6) and (7), are demonstrated below for some selected systems.

²The separate notations for perturbations, $\delta\mathbf{X}$ and $\delta\dot{\mathbf{X}}$, are used for different Eqs. (1) and (3) respectively.

³Taylor expansion of a differentiable \mathbf{F} can be done at any point in phase space.

⁴If the velocity vector, \mathbf{F} , is differentiable.

⁵These are possible only in nonlinear systems.

2.1. One-dimensional systems:

In order to demonstrate the properties of perpetual points we first consider a simple, analytically traceable, dynamical system

$$\dot{x} = x^2 + \alpha. \quad (8)$$

For $\alpha > 0$ there is no fixed point and hence system settles at infinity. The acceleration, $\ddot{x} = 2x(x^2 + \alpha)$, of this system has one perpetual point, $x_{PP} = 0$. The velocity at this point is α while eigenvalue, corresponding to $\mathbf{G}_{\mathbf{x}^T}|\mathbf{x}_{PP}=0$, is $\mu = 2\alpha$. The general solution of Eq. (5) for arbitrary initial perturbations $\delta x(0)$ and $\delta \dot{x}(0)$ around this perpetual point is

$$\delta x = c_{11} \exp(\sqrt{2\alpha}t) + c_{12} \exp(-\sqrt{2\alpha}t), \quad (9)$$

where $c_{11} = (\sqrt{2\alpha}\delta x(0) + \delta \dot{x}(0))/2\sqrt{2\alpha}$ and $c_{12} = (\sqrt{2\alpha}\delta x(0) - \delta \dot{x}(0))/2\sqrt{2\alpha}$. Here, $\delta \dot{x}(0) = 2x(0)\delta x(0)$ from Eq. (8). With simple manipulation of this solution and its derivative, we get the relation

$$2\alpha\delta x^2 - \delta \dot{x}^2 = [2\alpha - 4x(0)^2]\delta x(0)^2, \quad (10)$$

which clearly shows that both δx and $\delta \dot{x}$ decrease simultaneously as a trajectory approaches the PP i.e., in the neighborhood of this type of perpetual point where $\mu > 0$ system moves slowly. Therefore, this point is termed as slow perpetual point (SPP)⁶. The magnitude of velocity, $|\dot{x}|$, is shown in Fig. 1(a) while direction of motion in phase-space is indicated in (b). Here the size of the arrows indicate the magnitude of velocity. Note that the trajectory passes through the PP, in contrast to the FP where trajectories never cross.

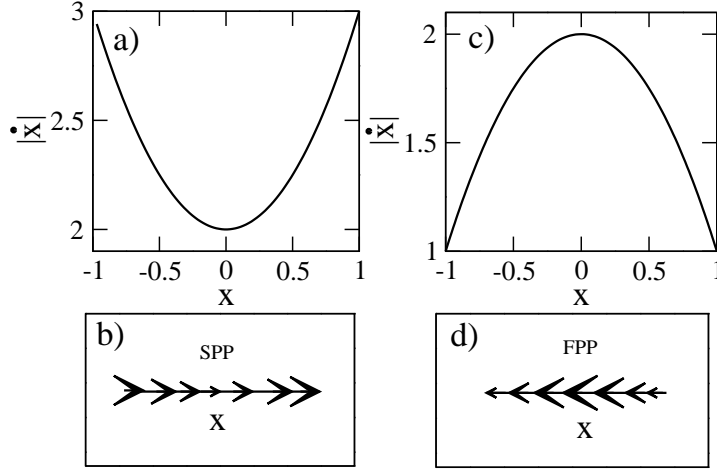


Fig. 1. Plots in top row show the magnitude of velocity while bottom row show the schematic motion in phase-space of Eq. (8). Left and right panels correspond to the parameter values $\alpha = 2$ and $\alpha = -2$, respectively.

However for $\alpha < 0$, Eq. (8) has two fixed points, $x_{FP} = -\sqrt{|\alpha|}$ and $\sqrt{|\alpha|}$, which are respectively stable and unstable i.e., initial conditions $x(0) < \sqrt{|\alpha|}$ lead to $x_{FP} = -\sqrt{|\alpha|}$ otherwise settle at infinity. The perpetual point for this case is $x_{PP} = 0$ where velocity and eigenvalue are $-|\alpha|$ and $\mu = -2|\alpha|$ respectively. The general solution of Eq. (5) around this PP is

$$\delta x = c_{11} \cos(\sqrt{2|\alpha|}t) + c_{12} \sin(\sqrt{2|\alpha|}t) \quad (11)$$

where $c_{11} = \delta x(0)$ and $c_{12} = \delta \dot{x}(0)/\sqrt{2|\alpha|} = 2x(0)\delta x(0)/\sqrt{2|\alpha|}$, which along with its derivatives give the relation

$$2|\alpha|\delta x^2 + \delta \dot{x}^2 = [2|\alpha| + 4x(0)^2]\delta x(0)^2. \quad (12)$$

⁶ The nomenclature “fast” or “slow” for perpetual points is used because system moves with “maximum” or “minimum” velocity at these points. This nomenclature is considered in the spirit of definition “stationary point” where velocity is zero.

This indicates that as δx decreases i.e., trajectory reaches to $x_{PP} = 0$, $\delta \dot{x}$ increases as shown in Figs. 1(c) and 1(d). This suggests that as the system reaches PP, its velocity becomes higher and higher, and hence it moves faster. Therefore this type of PP, where $\mu < 0$, is termed as faster perpetual point (FPP)⁶.

For special cases when eigenvalues are zero, Eq. (7) can be written as $\delta \mathbf{X} = \mathbf{V}_i[c_{i1} + c_{i2}t]$ where \mathbf{V}_i is eigenvector corresponding to the zero eigenvalue. For an example, consider system $\dot{x} = (x - 1)^3 - \alpha$, $\alpha > 0$ which has one unstable fixed point, $x_{FP} = 1 + \alpha^{1/3}$, with $\lambda = 3\alpha^{2/3}$ and one perpetual point, $x_{PP} = 1$, with $\mu = 0$. Since $\mu = 0$ therefore there is no extremum in velocity. However the trajectory starting from $1 < x(0) < 2$ approaches $x = 1$ with high velocity and then becomes constant at PP. Once it passes through PP, velocity increases again. This can be seen as an inflection point in velocity-distance plot, or also from the solution $\delta x = \delta x(0) + t\delta \dot{x}$.

We observed similar results for other one-dimensional systems also – see Eq. (16). The formalism, (Eq. (5)), which is verified here for 1D-systems, now can be extended to higher dimensional systems, as given below.

2.2. Two-dimensional systems:

Here we consider the celebrated Duffing oscillator [Ott, 1993]

$$\ddot{x} + \beta \dot{x} - \alpha x + x^3 = 0. \quad (13)$$

It has three fixed points at $x_{FP} = 0$ and $\pm\sqrt{\alpha}$ for $\alpha > 0$. The first point is unstable while the last two are stable, which are shown in Fig. 2(a) in $(x_1 = x, x_2 = \dot{x})$ plane with open and filled circles. The perpetual points of this system are $(-\sqrt{\alpha/3}, -\frac{2\alpha}{3\beta}\sqrt{\alpha/3})$ and $(\sqrt{\alpha/3}, \frac{2\alpha}{3\beta}\sqrt{\alpha/3})$ ⁷. The stability analysis suggests that eigenvalues for both the PPs are $\mu = (\beta^2 \pm \sqrt{\beta^4 + 16\alpha^2/3})/2$. Since the largest eigenvalue is positive for these PPs the velocities should be slower. This is confirmed in Fig. 2(b) where the magnitude of velocity, $|F| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$, along the velocity vector, $(1, 0)^T$ at one PP is plotted. This clearly shows that the velocity is minimum at this PP.

In another example of two-dimensional system we consider a system having coexisting attractors, a stable fixed point and a limit cycle,

$$\begin{aligned} \dot{r} &= r(r - 1)(2 - r) \\ \dot{\theta} &= \omega. \end{aligned} \quad (14)$$

Here $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \tan^{-1} x_2/x_1$ correspond to the radius and the phase of the system with frequency ω . The fixed point analysis of this system gives two coexisting attractors at $r = 0$ (stable fixed point) and $r = 2$ (limit cycle), and one unstable periodic orbit (UPO) at $r = 1$. These, stable and unstable features, are shown in Fig. 2(c) with solid and dotted lines respectively. The perpetual points of this system are, $r_{PP} = 1 \pm 1/\sqrt{3}$ which form closed loci, shown in Fig. 2(c) with red-dashed lines. The eigenvalues at perpetual points are $\mu = -6r_{PP}(1 - r_{PP})^2(2 - r_{PP})$. Since $r_{PP} < 2$, therefore $\mu < 0$, which implies that velocities have maxima at these points. The other two-dimensional systems are also considered in Sec. 5.1.

2.3. Three-dimensional systems:

The self-excited attractors, whose basins intersect with the neighborhood of the fixed points (e.g. Lorenz, Rossler, Chua, etc.) have been studied in detail [Ott, 1993; Strogatz, 1994; Jordan & Smith, 2009; Kuznetsov, 2004; Kaushal & Prashar 2008; Lakshmanan & Rajasekar, 2003; Arrowsmith & Place, 1990]. Very recently, a new type of attractors called hidden attractors, that don't intersect with the neighborhood of any FP have been reported [Kuznetsov *et al.* 2010; Leonov *et al.* 2011a; Leonov *et al.* 2011b; Leonov *et al.* 2013]. Due to the absence of unstable FP in its neighborhood these type of attractors are less tractable. Therefore, it is also difficult to understand their characteristic behavior [Kuznetsov *et al.* 2010; Kuznetsov *et al.* 2011; Leonov *et al.* 2010; Leonov and Kuznetsov 2011; Leonov *et al.* 2011a, Leonov

⁷Here PPs are in fact the inflection points of the potential $x^4/4 - \alpha x^2/2$, Eq. (13).

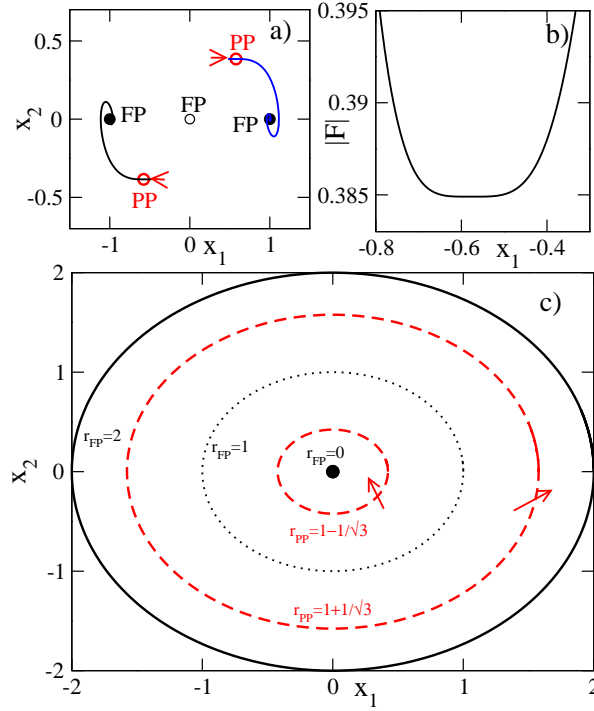


Fig. 2. The (a) dynamical behavior around the fixed and perpetual points, (b) the magnitude of velocity as a function of x_1 along the vector $(1,0)^T$ near one of the PPs of Eq. (13) for $\alpha = 1$ and $\beta = 1$. (c) The orbits and loci of fixed and perpetual points respectively of Eq. (14)-see text for details. Arrows show the directions of flow.

et al. 2011b; Leonov *et al.* 2011c; Chaudhuri & Prasad 2014]. As an example, we consider such a system that has no fixed point and provides hidden attractor [Wang & Chen, 2013],

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_2 + 3x_2^2 - x_1^2 - x_1x_3 + \alpha.\end{aligned}\tag{15}$$

This system doesn't have any fixed point for parameter $\alpha < 0$. A typical chaotic hidden attractor is shown in Fig. 3(a) for $\alpha = -0.05$. Because there is no fixed point in this system at this value of parameter, this attractor is termed as hidden one.

The perpetual points for this systems are $(0, x_{2PP\pm}, 0)$ where $x_{2PP\pm} = (1 \pm \sqrt{1 - 12\alpha})/6$. These are shown in Fig. 3(a) with red-circles. The largest eigenvalues μ_i , corresponding to the matrix $\mathbf{G}_{\mathbf{x}^T}$, at these PPs are $6x_{2PP\pm} - 1$ and $[5x_{2PP\pm} - 1 \pm \sqrt{13x_{2PP\pm}^2 - 14x_{2PP\pm} - 4\alpha + 1}]/2$ which are +ve for the both points⁸. Therefore the velocities at these points should be slower, which is confirmed in Fig. 3(b) where the magnitude of velocities are plotted along the velocity vectors at these points⁹. An another system with no fixed point is also considered in Sec. 5.2.

3. Perpetual Points vs. Fixed Points

In this section we present the comparative properties of PPs and FPs, as summarized in Table I: (i) The velocity at the perpetual points is either maximum or minimum or of inflection behavior while zero

⁸The eigenvalues, μ_i at x_{2PP+} and x_{2PP-} are $(1.2649, 0.4437 \pm i0.7470)$ and $(0.0684, -1.2649, -1.2892)$ respectively.

⁹ The velocity vectors at PP $(0, x_{2PP\pm}, 0)$ are $(x_{2PP\pm}, 0, -x_{2PP\pm} + x_{2PP\pm}^2 - \alpha)^T$. The velocities are calculated along the lines which are estimated using the parametric (p) form of the equation of a line passing through these PPs: $x_1 = px_{2PP\pm}$, $x_2 = x_{2PP\pm}$ and $x_3 = p(-x_{2PP\pm} + x_{2PP\pm}^2 + \alpha)$ which give as relation $x_3 = x_1(-x_{2PP\pm} + x_{2PP\pm}^2 + \alpha)/x_{2PP\pm}$.

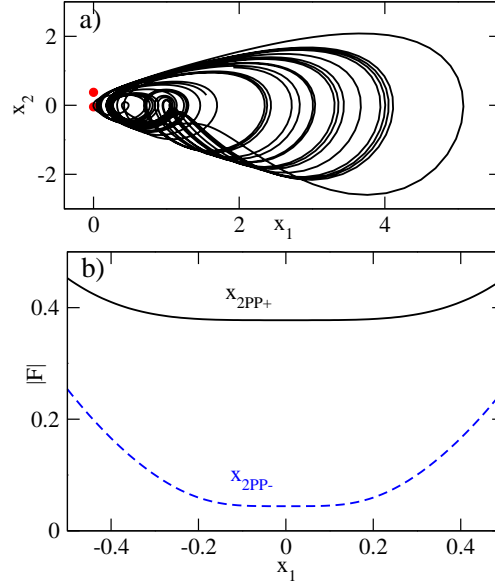


Fig. 3. (a) A hidden chaotic attractor and (b) the magnitude of velocities, along the velocity vectors passing through the perpetual points.

Table 1. Comparative properties of FP and PP. Here, μ_m is the largest nonzero eigenvalue among μ_i .

	Fixed Point	Perpetual Point
Velocity, $ F $	$ F = 0$	$ F \neq 0$ Extremum if $\mu_m \leq 0$ Inflection if $\mu_i = 0 \ \forall i$
Acceleration	0	0
l_i [Eq. (2)]	$+, -, 0$	At least one of $l_i = 0$
μ_i [Eq. (5)]	$\mu_i = \lambda_i^2$	$+, -, 0$
Motion of points	Stable or unstable periodic orbits	Closed locus
Trajectory	Moves towards or Moves away	Passes through

for fixed points. However, as per definition, acceleration is zero at both the PPs and FPs. (ii) The eigen values, λ_i , corresponding to Eq. (2) are either negative or positive or zero at fixed points. However one of l_i must be zero at PPs. Its reason can be explained as follows. In general, if $\mathbf{F}_{\mathbf{x}^T} \cdot \mathbf{F} = 0$ then there is either a trivial solution $\mathbf{F} = 0$ or infinite number of solutions. Since $\mathbf{F} \neq 0$ at the perpetual points therefore a finite number of solutions exists if and only if $\mathbf{F}_{\mathbf{x}^T}$ admits at least one zero eigenvalue corresponding to the eigenvector \mathbf{F} . (iii) Here, as we have observed in previous section, the value of μ_i could be either positive or negative or zero at PP. At FPs, first term in Eqs. (3) becomes zero due to $\mathbf{F} = \mathbf{0}$, and hence μ_i is related to l_i as $\mu_i = l_i^2$. (iv) A trajectory passes through a PP whereas it either moves away or settles on a fixed point. Note that in some cases, for example Eq. (14), PP forms a closed locus as shown in Fig. 2(c) whereas FP forms either stable or unstable periodic orbits.

4. Bifurcations

In order to explore the possibilities of occurrence of bifurcations of perpetual points as a function of parameters, say α , we first consider system Eq. (8). Shown in Fig. 4(a) are the filled and open circles for stable and unstable fixed points along with the solid-blue and dashed-red line for fast and slow PPs respectively. This shows that below $\alpha = 0$ there are two fixed points (stable and unstable) and one fast

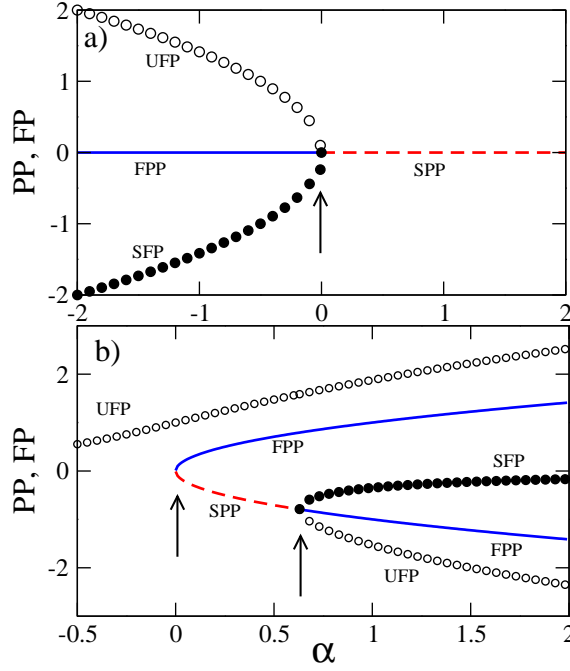


Fig. 4. The bifurcation diagrams of systems (a) Eq. (8) and (b) Eq. (16) as a function of parameter α . Arrows show the bifurcation points. Filled and open circles are the stable and unstable FPs while Solid-blue and dashed-red lines are FPP and SPP respectively.

perpetual point at $x = 0$ ¹⁰. As parameter α passes through $\alpha = 0$ there is no fixed point but the fast perpetual point becomes slower one i.e. there is change in magnitude of velocity from maximum to minimum across $\alpha = 0$. Therefore this change in dynamics may be termed as bifurcation of PP ($x_{PP} = 0$).

In another example, we construct a system,

$$\dot{x} = x^3 - 3\alpha x - 1. \quad (16)$$

Depending on the parameter, α , this system has either one or three fixed points which are shown in bifurcation diagram, Fig. 4(b). The open and filled circles correspond to the stable and unstable fixed points (denoted as SFP and UFP respectively). This system has two perpetual points, $x_{PP} = \pm\sqrt{\alpha}$, which are shown in Fig. 4(b) with solid and dashed lines. At $\alpha = 0$, the fast ($\mu = 6(-2\alpha^2 - \sqrt{\alpha})$) and the slow ($\mu = 6(-2\alpha^2 + \sqrt{\alpha})$) perpetual points (shown as solid and dashed lines respectively) are created. As parameter α is increased further, slow perpetual point becomes fast at $\alpha \sim 0.62$ where saddle-node bifurcation¹⁰ for fixed points occurs. We observe similar bifurcations in other systems (from Ref. [Strogatz, 1994]) also, which confirm that the perpetual points do show bifurcations behavior in parameters space.

5. Applications

5.1. Conservative vs. Dissipative

Checking a system whether it is conservative or not is an important problem in nonlinear dynamics. There are various ways to confirm whether a system is conservative or not [Strogatz, 1994; Ott, 1993]. In this work we show that this can be checked in a simple way by searching the perpetual points. Let us consider an autonomous Hamiltonian system $H = H(q, p)$ of one-degree of freedom (the following results can be extended for the systems with n -degrees of freedom also). Here q and p are the generalized co-ordinate and the momentum respectively. The equations of motion are

¹⁰In one-dimensional systems there must be at least one fast perpetual point in between two fixed points.

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial q}.\end{aligned}\tag{17}$$

The next higher order derivatives of these equations are given as

$$\begin{aligned}\ddot{q} &= \frac{\partial^2 H}{\partial p \partial q} \dot{q} + \frac{\partial^2 H}{\partial p^2} \dot{p} \\ \ddot{p} &= -\frac{\partial^2 H}{\partial q \partial q} \dot{q} - \frac{\partial^2 H}{\partial p \partial q} \dot{p},\end{aligned}\tag{18}$$

which can be recast in the matrix form as

$$\begin{bmatrix} \ddot{q} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 H}{\partial p \partial q} & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial q^2} & -\frac{\partial^2 H}{\partial p \partial q} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}\tag{19}$$

$$\Rightarrow \ddot{\mathbf{X}} = \mathbf{F}_{\mathbf{X}^T} \cdot \mathbf{F} \quad [\text{cf. Eq. (3)}]\tag{20}$$

where $\mathbf{X} = (q, p)^T$. For getting perpetual points we must have $\ddot{\mathbf{X}} = 0$. However, if $\mathbf{F}_{\mathbf{X}^T} \cdot \mathbf{F} = 0$ then there is either the trivial solution $\mathbf{F} = 0$ or there are infinite number of solutions. Since $\mathbf{F} \neq 0$ at the perpetual points, the finite solution exists if and only if $\mathbf{F}_{\mathbf{X}^T}$ admits at least one zero eigenvalue corresponding to the eigenvector \mathbf{F} . However, due to the special structure of the Hamiltonian, the trace of $\mathbf{F}_{\mathbf{X}^T}$ is zero. This means that the determinant of $\mathbf{F}_{\mathbf{X}^T}$ should also be zero. But $\mathbf{F}_{\mathbf{X}^T} = 0$ is possible only at the fixed point, hence the existence of PP is ruled out. This suggests that the existence of PP can confirm dissipation in the systems. This is demonstrated in the following examples.

Duffing systems: We first consider the Duffing system, Eq. (13), which has two perpetual points $(-\sqrt{\alpha/3}, -\frac{2\alpha}{3\beta}\sqrt{\alpha/3})$ and $(\sqrt{\alpha/3}, \frac{2\alpha}{3\beta}\sqrt{\alpha/3})$. These points are shown in Fig. 5(a) with squares. This clearly indicates that for zero strength of dissipation, $\beta \rightarrow 0$, the PP goes to infinity i.e. there is no PP in the conservative limit, and hence at $\beta \rightarrow 0$ system is conservative.

Plane pendulum: Consider the equation of a plane pendulum

$$\ddot{q} + \beta \dot{q} + \alpha \sin q = 0\tag{21}$$

where α and β are the system parameters. For $\beta = 0$, system is conservative and Hamiltonian (total energy) is constant of motion. Its equations of motion can be written as

$$\begin{aligned}\dot{q} &= p \\ \dot{p} &= \beta p - \alpha \sin q.\end{aligned}\tag{22}$$

The fixed points of this system are $(\pm k\pi, 0)$ where $k = \pm 1, \pm 2, \dots$. The perpetual points for this system are $(\pm(n + 1/2)\pi, \pm\alpha/\beta)$. These points, FPs (circles) and PPs (squares) are shown in Fig. 5(b). The color represents the acceleration $\sqrt{\ddot{q}^2 + \ddot{p}^2}$. Similar to the Duffing system, it clearly indicates that as strength of dissipation, $\beta \rightarrow 0$ the PP goes to infinity i.e. there is no PP in the conservative limit. Here also in strong dissipation, i.e. $\beta \rightarrow \infty$, $\dot{p}_{PP} \rightarrow 0$. Note that these PPs are at the inflection point of the potential of the Hamiltonian.

Lotka-Volterra system: We consider another celebrated system which is not in the Hamiltonian form. The governing equations of this system [Lotka, 1920] are

$$\begin{aligned}\dot{x} &= x(a - by) \\ \dot{y} &= y(cx - d)\end{aligned}\tag{23}$$

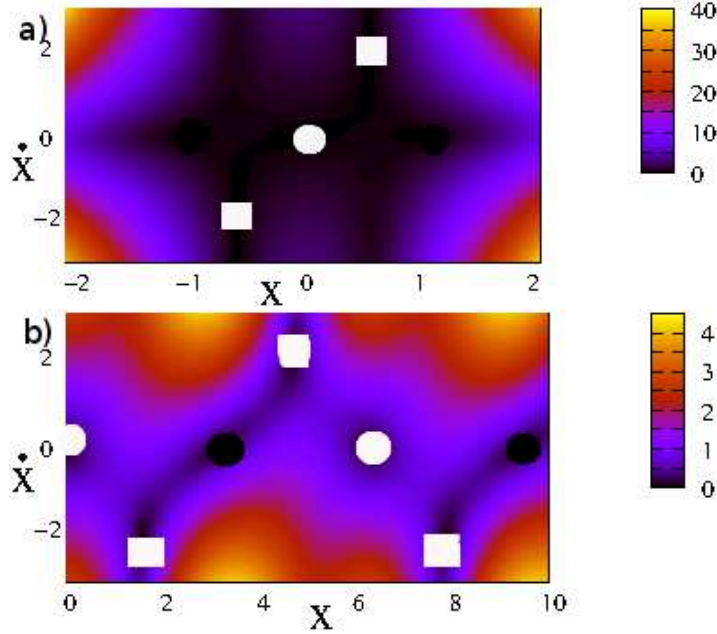


Fig. 5. Acceleration in $x - \dot{x}$ plane for (a) Duffing system, Eq. (13), and (b) plane pendulum, Eq. (21), at parameters $\alpha = 1$ & $\beta = 3$ and $\alpha = 2$ & $\beta = 1$ respectively.

where x and y are prey and predator populations respectively while a, b, c , and d are the system's parameters. Note that this system is quite different in structure as compared to the Duffing and Pendulum systems. However, after certain transformation [Lotka, 1920] it can be shown that this system has one invariant $I = d \log x - cx + a \log y - by$, suggesting that the system is conservative. For checking the existence of PP in this system, we examine the acceleration vector

$$[G(X)] = \begin{bmatrix} x(a - by)^2 - bxy(cx - d) \\ cxy(a - by) + y(cx - d)^2 \end{bmatrix}.$$

The fact is that $G(X) \neq 0$ for positive parameters. This implies that there are no perpetual points, and hence it is a conservative system. All these examples suggest that, in a system, if there exists a PP then it is dissipative otherwise conservative.

5.2. Locating hidden and coexisting attractors

The understanding of hidden attractors (cf. Sec. 2.3), as compared to the excitable ones, is difficult due to the absence of fixed points. Even to locate the hidden attractors in a given system, one requires proper search method consisting of analytical and numerical techniques [Leonov *et al.* 2010; Leonov *et al.* 2011b; Bragin *et al.* 2011]. In this work we show that we can use perpetual points to locate such hidden attractors.

Consider the system, described by Eq. (15), which has no fixed point but has one chaotic hidden attractor as shown in Fig. 3(a) (see Sec. 2). It has two perpetual points, $(0, x_{2PP+}, 0)$ and $(0, x_{2PP-}, 0)$ as shown in Fig. 6(a) with red-circles. The transient trajectories starting from these PPs are shown in Fig. 6(a). It clearly shows that the trajectories starting from the former one goes to the hidden attractor (Fig. 3(a)) where as the later one goes to infinity [Chaudhuri & Prasad 2014]. This confirms that the perpetual point, $(0, x_{2PP+}, 0)$, is useful to locate the hidden attractor.

We consider another example, the Nose-Hoover system [Jafari *et al.* 2013]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 x_3 \\ \dot{x}_3 &= x_2^2 - \alpha. \end{aligned} \tag{24}$$

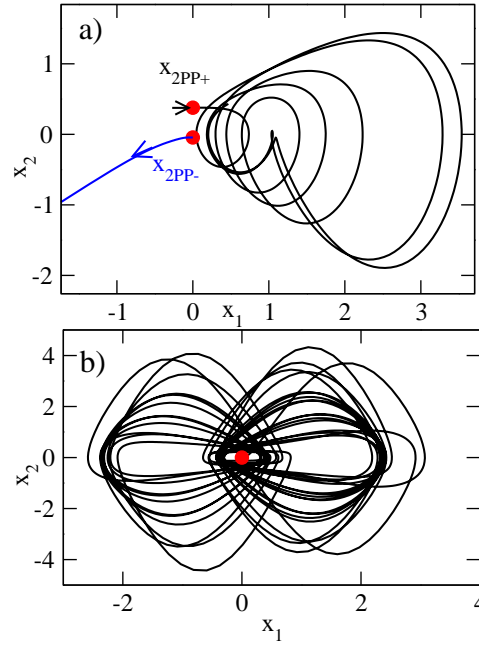


Fig. 6. The transient trajectories starting from perpetual points (red circles) for systems (a) Eq. (15) and (b) Eq. (24).

This system doesn't have any FP but shows chaotic motion ($\alpha = 1$), as shown in Fig. 6 (b). This system has one PP which is $(0, 0, 0)$. The trajectory starting from the perpetual point leads to the chaotic attractor. As in other similar systems (from Ref. [Jafari *et al.* 2013]) without FP, but having oscillations can also be located. These examples confirm that hidden attractor can be located with PP. These numerical results also indicate that hidden attractors (oscillating ones) also have reference point similar to the fixed points of self-excited oscillations. However, a detailed connection between the perpetual points and the hidden attractors needs to be established. Perhaps there is a need to develop a manifold type of theory for perpetual points as exists for fixed points [Kuznetsov 2004].

We observed that the coexisting attractors can also be located using perpetual points. We can easily see in Fig. 2(a) that the trajectories starting from different perpetual points go to different stable FPs, either $(0, -\sqrt{\alpha})$ or $(0, \sqrt{\alpha})$. The trajectories passing through the perpetual points are shown in Fig. 2(a) by black and blue lines. The conclusion that the PPs can be used to locate the coexisting attractors can also be drawn from Fig. 2(c) for limit cycle cases where initial conditions starting from the perpetual points $r_{PP} = 1 - 1/\sqrt{3}$ and $r_{PP} = 1 + 1/\sqrt{3}$ lead to the stable fixed point ($r_{FP} = 0$) and stable limit cycle ($r_{FP} = 2$) respectively. These examples suggest that the knowledge of perpetual points is useful for locating the hidden and coexisting attractors¹¹.

5.3. Transients

In phase space the velocity or the acceleration of a system doesn't remain constant (except for circular motion). The accelerations in the $x - \dot{x}$ plane for the systems described in Eqs. (13) and (21) are shown in Fig. 5 (color). The accelerations are different in different regions of phase space. Particularly at the PPs, accelerations are zero while velocities are extremum (Fig. 5). Therefore, as a trajectory moves in phase space the durations of travel for the equal distances are different. The variation of velocity also effects the transient time¹² e.g. If we start the system from an initial condition for which the trajectory passes

¹¹The multi-stable attractors [Pisarchik, & Feudel, 2014] where number of attractors are very large e.g. in conservative limit, are yet to be studied.

¹²Transient time is defined as the duration in which a system, starting from an initial condition, reaches to the asymptotic attractor.

through the neighborhood of perpetual points, where velocity is extremum (Fig. 5), then it requires either longer or shorter time (corresponding to SPP or FPP respectively) to reach the attractor. Therefore, in many applications where understanding of transients is important [Hastings, 2001; Ovaskainen & Hanski, 2002], location of PPs in phase space (Fig. 5) is useful.

6. Summary

Summary: A new set of points, termed as *perpetual points* where velocity remains nonzero while acceleration becomes zero, are observed. At these points the velocity can be either maximum or minimum or of inflection behavior. The properties of perpetual points, along with the fixed points, are summarized in Table I (Sec. 3).

The perpetual points, each having its extremum value of velocity, influence the transient dynamics. Therefore, the study of PPs is essential to understand many natural phenomena, e.g., experiments where transients persist for longer time, neural processing [Hastings, 2001; Ovaskainen & Hanski, 2002], transient chaos, and in ecology where the transient dynamics play an important role. Using these perpetual points one can also locate the presence of co-exiting as well as hidden attractors. The existence of PPs in a system can be used to confirm whether the system is conservative or not.

It is easier to examine the magnitude of the velocities in lower dimensions (particularly one and two dimensional systems). In case of high dimensional systems it is difficult to visualize the extrema of the velocities in phase space. Here, the presented generalized formulation (cf. Eqs. (4) and (6)) can easily be applied to any (lower or higher) dimensional systems. This work also shows that it may be useful to consider the higher derivatives (without any additional initial conditions) of the velocity vector for better understanding of nonlinear dynamical systems¹³. Here we have demonstrated the analysis for several different type of systems, which suggest that the presented results are quite general.

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¹³The higher-order derivatives of a nonlinear system may give either FPs/PPs only or an entirely new set of points. The physical meaning of the higher derivatives and the corresponding new points need to be explored.

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